Observer Design Tools for Nonlinear Flight Regimes

Harry G. Kwatny   Bor-Chin Chang

hkwatny@coe.drexel.edu   bchang@coe.drexel.edu

Department of Mechanical Engineering & Mechanics
Drexel University
32^{nd} & Chestnut Streets
Philadelphia, PA 19104

Abstract

Observer design for highly nonlinear dynamics is an important issue, particularly when the locally observable dynamics are not linearly observable. In such circumstances the ability to reduce the system to observable or observer form is key to observer design. We describe and illustrate symbolic computing tools to do that.

Keywords: Computer algebra, nonlinear observer, nonlinear systems.

1 Introduction

Observers are a central part of fault detection/identification processes as well as regulator approaches to reconfigurable control systems. Our immediate interests have to do with control reconfiguration in aircraft following actuator failure. When nonlinearities are essential, e.g., when operating around bifurcation points or near stability boundaries, issues of observability and observer design present new complexities that are absent in linear problems. For example, in linear systems, the input does not play a role in deciding observability. But a nonlinear system may be observable for some inputs and not so for others. As a result new theoretical paradigms for observer design for nonlinear systems have emerged over the past decade.

The literature contains many design alternatives for systems that are linearly observable. When this is not the case, available techniques are far more limited. Moreover, application experience from which to draw conclusions about their relative practical merits is virtually nonexistent. One reason for this is, undoubtedly, the lack of computational tools. In this paper we describe some results from an ongoing software development project. The ability to reduce the system to observable or observer form is fundamental to nonlinear observer design, and it is the main focus of this paper.

We begin with an overview of observability and a short and necessarily incomplete summary of nonlinear observer design methods. Then we describe the computations required to construct the observable and observer. Examples follow. Our results exploit the differential calculus constructions of [6] and extend the capabilities of the computations described by Bensacon and Bornard [7] to the multiple input/multiple output case.

2 Nonlinear System Observability and Observers

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i = f_u(x) \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in M$ (a neighborhood of x_0 in R^n), $u \in R^m$, and $y \in R^p$. We assume x_0 is an equilibrium point corresponding to zero input and output, i.e., $f(x_0) = 0$, $h(x_0) = 0$. The functions f, g_i, h are smooth.

2.1 Observability

The *observation space* $\mathcal{O}$ of system (1) is the linear space of functions $M \to R$ over the field R spanned by all functions of the form

$$\begin{gathered} L_{v_k} \cdots L_{v_1}(h_i),\ k \geq 0,\ 1 \leq i \leq p, \\ v_k, \cdots, v_1 \in \{f, g_1, \ldots, g_m\} \end{gathered} \tag{2}$$

It is important to emphasize that the observation space consists of all linear combinations of the functions (2) with real constant coefficients – viz., 'over the field R'. An analytic system (1) is observable on M if and only if for any $x_1, x_2 \in M$, $x_1 \neq x_2$, there is a function $\Phi \in \mathcal{O}$ such that $\Phi(x_1) \neq \Phi(x_2)$. Associated with the observation space $\mathcal{O}$ is it's differential $d\mathcal{O}$, the codistribution

$$d\mathcal{O} = \text{span}\,\{d\lambda \,|\, \lambda \in \mathcal{O}\}$$

The *observability codistribution*, Ω_O, is the smallest codistribution that contains the covectors

$\{dh_1, \ldots, dh_p\}$ and is invariant with respect to the vector fields $f, g_1, \ldots, g_m$. If $d\mathcal{O}$ is nonsingular, then $d\mathcal{O} = \Omega_O$.

Let us recall some common tests for observability of the nonlinear system (1). The system is locally observable at x_0 if the observability codistribution, Ω_O has rank n at x_0. This is called the *observability rank condition.* If x_0 is a regular point of $\Omega_O(x_0)$, the observability rank condition is necessary as well as sufficient. A sufficient condition for local observability at x_0 is that the somewhat more friendly distribution

$$\Omega_L = \operatorname{span}\left\{L_f^k\left(dh_i\right),\, 1 \le i \le p,\ 0 \le k \le n-1\right\}$$

has rank n at x_0. When $\dim \Omega_O(x_0) = n$ but $\dim \Omega_L(x_0) < n$, the implication is that some states are distinguishable only under the action of control inputs. When this occurs, most control inputs do distinguish the states. There are a few *singular inputs*, notably $u = 0$, that do not. Thus, when $\dim \Omega_L(x_0) = n$ we will use the terminology *observable for zero input at* x_0. It is also possible to test the linearization of (1) at x_0 for observability. That is, define

$$A_0 = \frac{\partial f}{\partial x}(x_0),\ C_0 = \frac{\partial h}{\partial x}(x_0)$$

and test the pair (A_0, C_0). If the linearization is observable then we say that it is *linearly observable at* x_0. Linear observability implies zero-input observability. It is easy to prove that a system is linearly observable at x_0 if and only if $\dim \Omega_L(x_0) = n$. Thus, we have the following hierarchy

$$\begin{array}{ccc}
\dim \Omega_O(x_0) = n & \Rightarrow & \textit{locally observable} \\
\Uparrow & & \Uparrow \\
\dim \Omega_L(x_0) = n & \Rightarrow & \textit{zero input observable} \\
\Updownarrow & & \Uparrow \\
\dim \begin{bmatrix} C_0 \\ C_0 A_0 \\ \vdots \\ C_0 A_0^{n-1} \end{bmatrix} = n & \Leftrightarrow & \textit{linearly observable}
\end{array}$$

2.2 Approaches to Nonlinear Observer Design

An observer for the system (1) is a dynamical system with inputs $y(\tau), u(\tau),\ 0 \le \tau \le t$ and output $\widehat{x}(t) \in R^n$ such that $\widehat{x}(t)$ is an estimate of $x(t)$ in the sense that $\|x(t) - \widehat{x}(t)\| \to 0$ as $t \to \infty$. When (1) is linearly observable (even detectable) there are many approaches to observer design. Most prevalent among these is the constant gain observer. Linear observability implies a matrix L can be chosen so that following is a local exponential observer for (1):

$$\dot{\widehat{x}} = f(\widehat{x}, u) + L\left(H\widehat{x} - y\right) \tag{3}$$

In fact, it is sometimes possible to choose L so that (3) is a global observer [10], [11], [12]. Other related approaches exist, like the sliding mode observer of [13].

Normal form observers are also limited to linearly observable (at least detectable) systems. Introduced in [14], they were generalized in [15] and [16]. Suppose $f(x)$ and $h(x)$ in (1) have formal power series expansions. Then one seeks a 'near identity' change of coordinates $x = T(z)$ in the form of a power series that leads to an (autonomous, $u = 0$) observer with linear error dynamics in the new coordinates.

On the other hand, if (1) is locally observable but not linearly observable, there are fewer options for observer design. Observer design based on linearization up to output injection was introduced in [1] for the single output case without inputs, extended to the multiple output case in [2], see also [9]. In this approach the idea is to transform the system (1) into the so-called 'observer form'

$$\dot{z} = Az + \varphi(y),\ y = Cz \tag{4}$$

where A, C is an observable pair. When this is done, observer design is very easy. As might be expected, systems that can be transformed into the form (4) are rare but it is interesting to note that linear observability is not necessary if we do not insist that the transformation be a diffeomorphism.

Xia and Zeitz [8] give conditions for observer construction for systems that are zero-input observable (see above hierarchy). This method (as do many others) begins with reduction of (1) to an 'observable form' that we will discuss below.

If the system (1) is locally observable, but not zero-input observable, then we have the approach of Hammouri et al [3-5]. The idea is to transform the system into the 'time varying' version of (4), specifically the 'observer form' given in (6), below.

3 Computational Tools

When (1) is not linearly observable, but nonetheless locally observable, we need to be able to reduce the system to either observable or observer form as a first step to observer design using existing methods. Even for linearly observable systems this may be a convenience. In this section we describe the computations needed to do that.

3.1 Control Sequences

One characterization of the observation space is given by the following result.

Lemma 3.1 *The observation space $\mathcal{O}$ is equivalent to the linear vector space of functions $M \to R$ over the field R*

$$\widetilde{\mathcal{O}} = \operatorname{span}_R \left\{ L_{f_{u^k}} \ldots L_{f_{u^1}}(h_i) \;\middle|\; \begin{array}{c} 1 \le i \le p,\ k \ge 0, \\ u^1, \ldots, u^k \in \{0,1\}^m \end{array} \right\}$$

Define a sequence of codistributions

$$\begin{aligned}&\mathcal{E}_0 := \text{span}\{dh\}\\ &\mathcal{E}_k = \mathcal{E}_{k-1}\\ &\quad +\text{span}\left\{dL_{f_{u_k}}\cdots L_{f_{u_1}}(h)\,|u_i\in\{0,1\}^m,\ i=1,\ldots,k\right\}\end{aligned}$$

We assume that there exists a smallest p^* such that

$$\mathcal{E}_0 \subset \cdots \subset \mathcal{E}_{p^*} = \mathcal{E}_{p^*+1} = d\mathcal{O}$$

Let n_k denote the codimension of $\mathcal{E}_{k-1}$ in $\mathcal{E}_k$. Then there exists sets of control sequences (see [4])

$$\begin{gathered}I_1 = \{(u_{i_1})\,|u_{i_1}\in\{0,1\}^m\},\\ I_2 = \{(u_{i_1},u_{i_2})\,|u_{i_1}\in\{0,1\}^m,\ u_{i_2}\in\{0,1\}^m\},\\ \vdots\end{gathered}$$

that satisfy

(a) If $(u_{i_1},\ldots,u_{i_j}) \in I_j$ then $(u_{i_1},\ldots,u_{i_{j-1}}) \in I_{j-1}$, for $j \geq 2$.

(b) The one forms

$$\bigcup_{l=1}^{k}\left\{dL_{f_{u_l}}\cdots L_{f_{u_1}}(h)\,|(u_{i_1},\ldots,u_{i_l})\in I_l\right\}\cup\{dh\}$$

form a basis for $\mathcal{E}_k$ on a neighborhood of x_0. The cardinal number of I_k is n_k.

We obtain the control sequences, I_k, by directly constructing the codistributions $\mathcal{E}_k$, sequentially. See [4] for more details about the single output case.

3.2 Observability Indices

Consider the set of I_j consisting of n_j j-tuples:

$$I_j = \left\{\left(u^1_{i_1},\ldots,u^1_{i_j}\right),\ldots,\left(u^{n_j}_{i_1},\ldots,u^{n_j}_{i_j}\right)\right\}$$

and define the $p\cdot n_j$ -vector of j^{th} order Lie derivatives

$$L_{f_{t_j}}(h) = \begin{bmatrix} L_{f_{u^1_{i_1}}}\cdots L_{f_{u^1_{i_j}}}(h)\\ \vdots \\ L_{f_{u^{n_j}_{i_1}}}\cdots L_{f_{u^{n_j}_{i_j}}}(h)\end{bmatrix}$$

Now, consider the collection of covectors $dL_{f_{t_i}}(h)$ for $i = 0,\ldots,p^*$, which we can arrange in the (block) tableau

$$\begin{matrix} dh_1 & dh_2 & \cdots & dh_p\\ dL_{f_{t_1}}(h_1) & dL_{f_{t_1}}(h_2) & \cdots & dL_{f_{t_1}}(h_p)\\ \vdots & \vdots & \vdots & \vdots\\ dL_{f_{t_{p^*}}}(h_1) & dL_{f_{t_{p^*}}}(h_2) & \cdots & dL_{f_{t_{p^*}}}(h_p)\end{matrix}$$

From this set we seek to identify a maximal set of independent covectors. We can do this by searching down columns or across rows (recall the linear counterpart). For a row search, begin with the first row and work from left to right, then move to the next row. If the outputs are them selves independent, we identify p chains of covectors $dh_i \quad dL_{f^1_i}(h_i) \quad \cdots \quad dL_{f_i^{\kappa_1-1}}(h_i)$ of length κ_i, $i = 1,\ldots,p$. The integers κ_i are the *observability indices.* For an observable system $\kappa_1+\kappa_2+\cdots+\kappa_p = n$.

3.3 Observable, Observer Forms

If the system is observable, then we can define new state variables $z \in R^n$ via the transformation $x \to z$.

$$z = \begin{bmatrix} h_1\\ \vdots\\ L_{f_1^{k_1-1}}(h_1)\\ \vdots\\ h_p\\ \vdots\\ L_{f_p^{k_p-1}}(h_p)\end{bmatrix} \tag{5}$$

If the inverse is continuous and the the transformed equations produce unique solutions we call the transformed equations an *observable form.* This is consistent with the usual terminology for linear systems and autonomous nonlinear systems. In the latter case, the transformed equations are in the form of p chains,

$$\begin{matrix} \dot z_1 = z_2 & \cdots & \dot z_{\kappa_1+\cdots+\kappa_{p-1}+1} = z_{\kappa_1+\cdots+\kappa_{p-1}+2}\\ \vdots & \cdots & \vdots\\ \dot z_{\kappa_1-1} = z_{\kappa_1} & \cdots & \dot z_{\kappa_1+\cdots+\kappa_p-1} = z_{\kappa_1+\cdots+\kappa_p}\\ \dot z_{\kappa_1} = \varphi_1(z) & \cdots & \dot z_{\kappa_1+\cdots+\kappa_p} = \varphi_p(z)\\ y_1 = z_1 & \cdots & y_p = z_{\kappa_1+\cdots+\kappa_{p-1}+1}\end{matrix}$$

Remark 3.2 (Xia and Zeitz) *Note that if*

$$\text{rank}\begin{bmatrix} dh_1\\ \vdots\\ dL_{f_1^{k_1-1}}(h_1)\\ \vdots\\ dh_p\\ \vdots\\ dL_{f_p^{k_p-1}}(h_p)\end{bmatrix}(x_0) = n$$

the implicit function theorem guarantees the existence of a smooth (local) inverse of the transformation (5) so that the transformation is a diffeomorphism. However, an inverse may exist even if the rank condition fails. In this case, the inverse will only be continuous. If the transformed differential equations have unique solutions on a neighborhood of x_0, then this is still a useful transformation. This point is described more fully in Xia and Zeitz [8].

Now, we consider transforming (1) into the special form (time-varying linear up to output injection)

$$\begin{gathered}\dot z = A(u(t))\,z + \varphi(y,u(t))\\ y = Cz\end{gathered} \tag{6}$$

In this form it is possible to use linear methods for observer design. Equation (6) will be called an *observer from* of which (4) is a special case. Not every locally observable nonlinear system (1) has an observer form.

The formulation we follow is that of [3-5]. First, let us introduce some definitions. Consider a set of p vector fields, $X = \{X_1, \ldots, X_p\}$. Sequentially define sets of $p+1$-forms

$$\begin{array}{l} \Omega_1^X = \mathrm{span}_R \left\{ dL_{f_u}(h) \wedge dh \,|u \in \{0,1\}^m \right\} \\ \Omega_{k+1}^X = \mathrm{span}_R \left\{ dL_{f_u}(i_X \alpha) \wedge dh \left| \alpha \in \Omega_k^X, \; u \in \{0,1\}^m \right. \right\} \\ \Omega^X = \sum_{k \geq 1} \Omega_k^X \end{array}$$

Let $i_f(\omega)$ denote the usual contraction of the form ω with respect to the vector field f. The we use the notation

$$i_X(\omega) = i_{X_1} \circ \cdots \circ i_{X_p}(\omega)$$

The following is multiple output version of a theorem of Hammouri and Kinnaert [4] (see also [5]).

Proposition 3.3 *The system (1) is transformable into the observer form (6) if and only if:*

(1) $dh_1 \wedge \cdots \wedge dh_p(x_0) \neq 0$

(2) The set of vector fields $X_1, \ldots, X_p$ *satisfies*

- *(a)* $\dim \Omega^X = n - p$
- *(b)* $\forall \omega \in \Omega^X, \; d i_X(\omega) = 0$
- *(c)* $i_X(\omega_1) \wedge \cdots \wedge i_X(\omega_{n-p}) \wedge dh_1 \wedge \cdots \wedge dh_p(x_0) \neq 0$

If these conditions hold, then the transformation is given by

$$\begin{array}{l} z_1 = h_1(x), \ldots, z_p = h_p(x) \\ dz_{j+p} = i_X(\omega_j), \quad j = 1, \ldots, n-p \end{array}$$

where $\omega_j), ; j = 1, \ldots, n-p$ *is a basis for* Ω^X.

Now, we need to provide a construction for the set of vector fields X. First, obtain a set of vector fields $Y_1, \ldots, Y_p$ that satisfy

$$\begin{bmatrix} dh_1 \\ \vdots \\ dL_{f_1^{k_1-1}}(h_1) \\ \vdots \\ dh_p \\ \vdots \\ dL_{f_p^{k_p-1}}(h_p) \end{bmatrix} \begin{bmatrix} Y_1 & \cdots & Y_p \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 & & 0 \\ 0 & & \vdots \\ 0 & \cdots & \vdots \\ \vdots & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}$$

For any control sequence $u_1, u_2, \ldots$ we can define the set of vector fields

$$Z^i_{u_1 \cdots u_{\kappa_i - 1}} = \left[f_{u_{\kappa_i - 1}}, \left[\cdots \left[f_{u_1}, Y_i \right] \cdots \right] \right], \; i = 1, \ldots, p$$

Now, identify the subset of control sequences $\mathcal{I} \subseteq I_{p^*}$ that satisfy

$$\det \left[L_{Z^1_{u_1 \cdots u_{\kappa_1 - 1}}}(h) \; \cdots \; L_{Z^p_{u_1 \cdots u_{\kappa_p - 1}}}(h) \right] \neq 0$$

and use any one of these sequences to obtain

$$\begin{array}{l} \begin{bmatrix} X_1 & \cdots & X_p \end{bmatrix} = \\ \quad \begin{bmatrix} L_{Z^1_{u_1 \cdots u_{\kappa_1 - 1}}}(h) & \cdots & L_{Z^p_{u_1 \cdots u_{\kappa_p - 1}}}(h) \end{bmatrix}^{-1}. \\ \qquad \begin{bmatrix} Z^1_{u_1 \cdots u_{\kappa_1 - 1}} & \cdots & Z^p_{u_1 \cdots u_{\kappa_p - 1}} \end{bmatrix} \end{array}$$

3.4 Implementation

The computations described above have been implemented in a *Mathematica* package. The package has three primary functions:

1. ObservabilityIndices
2. ObservableTransform
3. LinearizeToOutputInjection

These are supported by a number of utility functions that compute the control sequences, solve the first order partial differential equations of Proposition (3.3), and others. Underlying these calculations are basic tools for working with differential forms. We have slightly extended the Exterior Differential Calculus package of Bonanos [6]

4 Examples

Four examples follow that illustrate the computations. In each case we compute both the observable and observer forms. Example (4.4) is zero-output observable but not linearly observable. Example (4.5) is locally observable but does not satisfy the observability rank condition. Example (4.6) is linearly observable. Example (4.7) is not zero-input observable but satisfies the observablity rank condition.

Example 4.4 (Xia & Zeitz example 2) *First, we consider a simple two output, autonomous example from [8]. Although the transformation is smooth, its inverse is only continuous.*

```
In[1]:=  f = {x1, x2};
         h = {x1^3 + x2^5};
         x = {x1, x2};

In[2]:=  ObservabilityIndices[f, h, x, {u}]
Out[2]=  {2}
```

observable form

```
In[3]:=  {Trans, Ind} = ObservableTransform[f,
             h, x, {u}]
Out[3]=  {{x1^3 + x2^5, 3 x1^3 + 5 x2^5}, {2}}

In[4]:=  z = {z1, z2};

In[5]:=  InvTrans = InverseTransformation[x, z, Trans]
```

Out[5]= $\Big\{-\Big(-\frac{1}{2}\Big)^{1/3}(5\,\mathrm{z1}-\mathrm{z2})^{1/3},$
$-\Big(-\frac{1}{2}\Big)^{1/5}(-3\,\mathrm{z1}+\mathrm{z2})^{1/5}\Big\}$

In[6]:= $\{newf, newg, newh\} = TransformSystem[f/.u \to 0, Coefficient[f,u], h, x, z, Trans, InvTrans]$

Out[6]= $\{\{\mathrm{z2}, -15\,\mathrm{z1}+8\,\mathrm{z2}\}, \{0,0\}, \{\mathrm{z1}\}\}$

Observer form

In[7]:= $Trans = LinearizeToOutputInjection[f, h, x, \{u\}]$

Out[7]= $\{\mathrm{x1}^3+\mathrm{x2}^5, 5\,\mathrm{x1}^3+3\,\mathrm{x2}^5\}$

In[8]:= $z = \{z1, z2\};$

In[9]:= $InvTrans = InverseTransformation[x, z, Trans]$

Out[9]= $\Big\{-\Big(-\frac{1}{2}\Big)^{1/3}(-3\,\mathrm{z1}+\mathrm{z2})^{1/3},$
$-\Big(-\frac{1}{2}\Big)^{1/5}(5\,\mathrm{z1}-\mathrm{z2})^{1/5}\Big\}$

In[10]:= $\{newf, newg, newh\} = TransformSystem[f/.u \to 0, Coefficient[f,u], h, x, z, Trans, InvTrans]$

Out[10]= $\{\{8\,\mathrm{z1}-\mathrm{z2}, 15\,\mathrm{z1}\}, \{0,0\}, \{\mathrm{z1}\}\}$

Example 4.5 (Xia & Zeitz example 3) *Now consider a nonautonomous example, also from [8]. It is not zero-input observable. Again the inverse transformations are not smooth.*

In[11]:= $p = 3;$
$f = \{x2\hat{\ }p, x2\,u\};$
$h = \{x1\};$
$x = \{x1, x2\};$

In[12]:= $ObservabilityIndices[f, h, x, \{u\}]$

Out[12]= $\{2\}$

Observable form

In[13]:= $\{Trans, Ind\} = ObservableTransform[f, h, x, \{u\}]$

Out[13]= $\{\{\mathrm{x1}, \mathrm{x2}^3\}, \{2\}\}$

In[14]:= $z = \{z1, z2\};$

In[15]:= $InvTrans = InverseTransformation[x, z, Trans]$

Out[15]= $\{\mathrm{z1}, \mathrm{z2}^{1/3}\}$

In[16]:= $\{newf, newg, newh\} = TransformSystem[f/.u \to 0, Coefficient[f,u], h, x, z, Trans, InvTrans]$

Out[16]= $\{\{\mathrm{z2}, 0\}, \{0, 3\,\mathrm{z2}\}, \{\mathrm{z1}\}\}$

Observer form

In[17]:= $Trans = LinearizeToOutputInjection[f, h, x, \{u\}]$

Out[17]= $\{\mathrm{x1}, -\mathrm{x2}^3\}$

In[18]:= $z = \{z1, z2\};$

In[19]:= $InvTrans = InverseTransformation[x, z, Trans]$

Out[19]= $\{\mathrm{z1}, -\mathrm{z2}^{1/3}\}$

In[20]:= $\{newf, newg, newh\} = TransformSystem[f/.u \to 0, Coefficient[f,u], h, x, z, Trans, InvTrans]$

Out[20]= $\{\{-\mathrm{z2}, 0\}, \{0, 3\,\mathrm{z2}\}, \{\mathrm{z1}\}\}$

Example 4.6 (Hou and Pugh) *This example is from Hou and Pugh [9]. They propose a method for linearization to output injection for multiple output autonomous systems different from that implemented here. To obtain the observer form we need to reorder the outputs.*

In[21]:= $f = \{x2, x3\,x2, x2\};$
$CC = \{\{0,1\},\{1,0\}\};$
$h = CC.\{x1, x3\};$
$x = \{x1, x2, x3\};$

In[22]:= $ObservabilityIndices[f, h, x, \{u\}]$

Out[22]= $\{2, 1\}$

Observable Form

In[23]:= $\{Trans, Ind\} = ObservableTransform[f, h, x, \{u\}]$

Out[23]= $\{\{\mathrm{x3}, \mathrm{x2}, \mathrm{x1}\}, \{2, 1\}\}$

In[24]:= $z = \{z1, z2, z3\};$

In[25]:= $InvTrans = InverseTransformation[x, z, Trans]$

Out[25]= $\{\mathrm{z3}, \mathrm{z2}, \mathrm{z1}\}$

In[26]:= $\{newf, newg, newh\} = TransformSystem[f/.u \to 0, Coefficient[f,u], h, x, z, Trans, InvTrans]$

Out[26]= $\{\{\mathrm{z2}, \mathrm{z1}\,\mathrm{z2}, \mathrm{z2}\}, \{0,0,0\}, \{\mathrm{z1}, \mathrm{z3}\}\}$

Observer Form

In[27]:= $Trans = LinearizeToOutputInjection[f, h, x, \{u\}]$

Out[27]= $\{\mathrm{x3}, \mathrm{x1}, -2\,\mathrm{x2}+\mathrm{x3}^2\}$

In[28]:= $z = \{z1, z2, z3\};$

$InvTrans = InverseTransformation[x, z, Trans]$

Out[28]= $\Big\{\mathrm{z2}, \frac{1}{2}(\mathrm{z1}^2-\mathrm{z3}), \mathrm{z1}\Big\}$

In[29]:= $\{newf, newg, newh\} = TransformSystem[f/.u \to 0, Coefficient[f,u], h, x, z, Trans, InvTrans]$

Out[29]= $\Big\{\Big\{\frac{1}{2}(\mathrm{z1}^2-\mathrm{z3}), \frac{1}{2}(\mathrm{z1}^2-\mathrm{z3}), 0\Big\},$
$\{0,0,0\}, \{\mathrm{z1}, \mathrm{z2}\}\Big\}$

Example 4.7 *Now we consider a more elaborate example. The system is locally observable, but it is not observable with zero input.*

In[30]:= $f = \{Exp[x1+x2]-1+ux1\hat{\ }2,$
$-Exp[x1+x2]+1+u(Exp[x3-x2]$
$-Exp[-x1-x2]-x1\hat{\ }2),$
$-Exp[x1+x2]+1+x1\hat{\ }3Exp[-x1-x3]-ux1\hat{\ }2,$
$x5, x1\};$
$h = \{x1, x4\};$
$x = \{x1, x2, x3, x4, x5\};$

In[31]:= $ObservabilityIndices[f, h, x, \{u\}]$

Out[31]= $\{3, 2\}$

Observable form

In[32]:= $\{Trans, Ind\} = ObservableTransform[f, h, x, \{u\}]$

Out[32]= $\{\{x1, -1 + e^{x1+x2}, e^{x1+x2}(1 - e^{-x1-x2} - e^{x1+x2} + e^{-x2+x3} - x1^2) + e^{x1+x2}(-1 + e^{x1+x2} + x1^2), x4, x5\}, \{3, 2\}\}$

In[33]:= $z = \{z1, z2, z3, z4, z5\};$

In[34]:= $InvTrans = InverseTransformation[x, z, Trans]$

Solve :: ifun : Inverse functions are being used by Solve, so some solutions may not be found.

Out[34]= $\{z1, -z1 + \mathrm{Log}[1 + z2], -z1 + \mathrm{Log}[1 + z3], z4, z5\}$

In[35]:= $\{newf, newg, newh\} = TransformSystem[f/.u \to 0, Coefficient[f, u], h, x, z, Trans, InvTrans]$

Out[35]= $\{\{z2, 0, z1^3, z5, z1\}, \{z1^2, z3, 0, 0, 0\}, \{z1, z4\}\}$

Observer form

In[36]:= $Trans = LinearizeToOutputInjection[f, h, x, \{u\}]$

Out[36]= $\{x1, x4, -e^{x1+x2}, -x5, e^{x1+x3}\}$

In[37]:= $z = \{z1, z2, z3, z4, z5\};$

In[38]:= $InvTrans = InverseTransformation[x, z, Trans]$

Solve :: ifun : Inverse functions are being used by Solve, so some solutions may not be found.

Out[38]= $\{z1, -z1 + \mathrm{Log}[-z3], -z1 + \mathrm{Log}[z5], z2, -z4\}$

In[39]:= $\{newf, newg, newh\} = TransformSystem[f/.u \to 0, Coefficient[f, u], h, x, z, Trans, InvTrans]$

Out[39]= $\{\{-1 - z3, -z4, 0, -z1, z1^3\}, \{z1^2, 0, 1 - z5, 0, 0\}, \{z1, z2\}\}$

5 Conclusions

We have described symbolic computations for reducing a nonlinear smooth affine systems to observable and observer forms, when possible, as the first step in observer design. These tools can be applied to systems that are linearly observable, locally observable with zero input or merely locally observable.

6 References

1. A. J. Krener and A. Isidori, "Linearization by Output Injection and Nonlinear Observers," Systems and Control Letters, vol. 3, pp. 47-52, 1983.

2. A. J. Krener and W. Respondek, "Nonlinear Observers with Linearizable Error Dynamics," SIAM Journal of Control and Optimization, vol. 23, pp. 197-216, 1985.

3. H. Hammouri and J. P. Gauthier, "Bilinearization up to Output Injection," Systems and Control Letters, vol. 11, pp. 139-149, 1988.

4. H. Hammouri and M. Kinnaert, "A New Procedure for Time-Varying Linearization up to Output Injection," Systems and Control Letters, vol. 28, pp. 151-157, 1996.

5. H. Hammouri and J. P. Gauthier, "Time Varying Linearization up to Output Injection," presented at 28th Conference on Decision and Control, Tampa, Florida, 1999.

6. S. Bonanos, "Exterior Differential Calculus," Mathematica Notebook, www.mathsource.com/ Content/ Enhancements/ Algebraic/ 0210-935

7. G. Besancon and G. Bornard, "Symbolic Computation for Nonlinear Systems Analysis: the Example of Bilinearization," Mathematics and Computers in Simulation, vol. 44, pp. 11-27, 1997.

8. X. Xia and M. Zeitz, "On Nonlinear Continuous Observers," International Journal of Control, vol. 66, pp. 943-954, 1997.

9. M. Hou and A. C. Pugh, "Observer with Linear Error Dynamics for Nonlinear Multi-Output Systems," Systems and Control Letters, vol. 37, pp. 1-9, 1999.

10. S. Raghavan and J. K. Hedrick, "Observer Design for a Class of Nonlinear Systems," International Journal of Control, vol. 59, pp. 515-528, 1994.

11. F. E. Thau, "Observing the State of Nonlinear Dynamical Systems," International Journal of Control, vol. 17, pp. 471-479, 1973.

12. R. Rajamani, "Observers for Lipschitz Nonlinear Systems," IEEE Transactions on Automatic Control, vol. 43, pp. 397-401, 1998.

13. B. L. Walcott and S. H. Zak, "State Observation of Nonlinear Uncertain Dynamical Systems," IEEE Transactions on Automatic Control, vol. 32, pp. 166-170, 1987.

14. N. Kazantzis and C. Kravaris, "Nonlinear Observer Design Using Lyapunov's Auxiliary Theorem," Systems and Control Letters, vol. 34, pp. 241-247, 1998.

15. A. J. Krener and M. Xiao, "Nonlinear Obsrver Design in the Siegal Domain Through Coordinate Changes," Proceedings of NOLCOS, St. Petersburg, 2001.

16. H. G. Kwatny and B. C. Chang, "Exponential Observers via Normal Form Constructions." submitted, 2001.